Gaussian paradox and clustering in intermittent turbulent signals

A. Bershadskii

ICAR, P.O. Box 31155, Jerusalem 91000, Israel

A relation between intermittency and clustering phenomena in velocity field has been revealed for homogeneous fluid turbulence. It is described how the intermittency exponent can be split into sum of two other exponents. One of these exponents (cluster-exponent) characterizes clustering of the 'zero'-crossing points in nearly Gaussian velocity field and another exponent is related to the tails of the velocity probability distribution. The cluster-exponent is uniquely determined by the energy spectrum of the nearly Gaussian velocity field and entire dependence of the intermittency exponent on Reynolds number is determined by the cluster-exponent.

PACS numbers: 47.27.-i, 47.27.Gs

Relation between intermittency of dissipation field [1],[2],[3] and clustering in turbulent velocity field [4, 5] is still very obscure. Taking into account that turbulent velocity itself is nearly Gaussian [6] the clustering phenomenon in the velocity field should be also a 'Gaussian' one, while the intermittency phenomenon is usually associated just with non-Gaussian properties of the velocity derivatives [1],[2],[6]. On the other hand, it is expected that high frequency events in the velocity field should provide significant contribution to the turbulent dissipation and, especially, to its high order moments [1],[7],[8]. Therefore, to find relationship between the two phenomena is rather a non-trivial task that demands special tools. In our recent papers [4],[5] we introduced a cluster-exponent to describe quantitatively clustering in turbulent velocity field and we found an empirical relationship between the cluster- and intermittency exponents. In present paper we will describe how the intermittency exponent
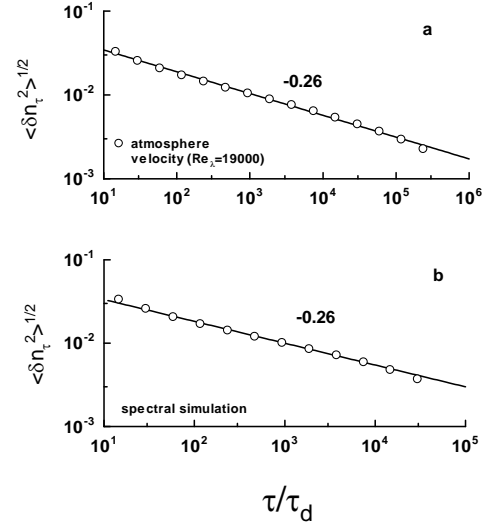

FIG. 2: As in Fig. 1 but for atmospheric surface layer at $Re_\lambda = 19000$ [12] (Fig. 2a) and for its Gaussian spectral simulation (Fig. 2b).
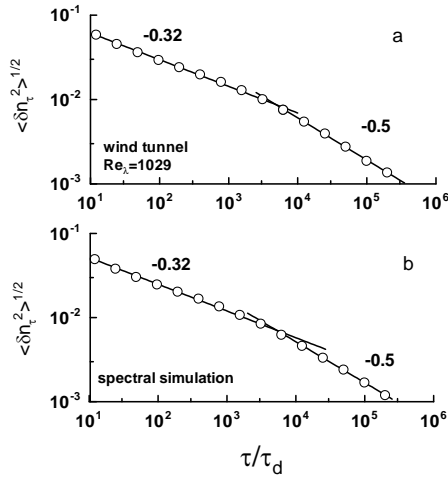

FIG. 1: Standard deviation of the running density fluctuations against τ for for the velocity fluctuations measured in the wind-tunnel [11] (Fig. 1a) and for their Gaussian spectral simulation (Fig. 1b).

can be split into sum of two other exponents (Eq. 6). One of these exponents is the cluster-exponent [4] and another exponent is related to the tails of the velocity probability distribution. We also will show that the cluster-exponent is uniquely determined by the energy spectrum of the nearly Gaussian velocity field. This simple splitting of the intermittency exponent on the Gaussian and 'tail' components can be considered as a solution of the problem which was formulated above. Moreover, for finite Reynolds numbers Re_λ the energy spectrum depends on Re_λ and, therefore, the cluster-exponents depends on Re_λ, while the 'tail' component of the intermittency exponent is independent on Re_λ. Therefore, the entire dependence of the intermittency exponent on Re_λ is uniquely determined by its Gaussian component (through the clustering phenomenon) that we will call Gaussian paradox.

Let us count the number of 'zero'-crossing points of the signal in a time interval τ and consider their running

density n_τ. Let us denote fluctuations of the running density as $\delta n_\tau = n_\tau - \langle n_\tau \rangle$, where the brackets mean the average over long times. We will be interested in the scaling variation of the standard deviation of the running density fluctuations $\langle \delta n_\tau^2 \rangle^{1/2}$ with τ

$$\langle \delta n_\tau^2 \rangle^{1/2} \sim \tau^{-\alpha_n} \qquad (1)$$

For white noise it can be derived analytically [9],[10] that $\alpha_n = 1/2$ (see also [4]). In Figure 1a and 2a we show calculations of the standard deviation for a turbulent velocity signal obtained in the wind-tunnel experiment [11] and for a velocity signal obtained in an atmospheric experiment [12] respectively. The straight lines are drawn in the figures to indicate scaling (1). One can see two scaling intervals in Fig. 1. The left scaling interval covers both dissipative and inertial ranges, while the right scaling interval covers scales larger then the integral scale of the flow. While the right scaling interval is rather trivial (with $\alpha_n = 1/2$, i.e. without clustering), the scaling in the left interval (with $\alpha_n < 1/2$) indicates clustering of the high frequency fluctuations. The cluster-exponent α_n decreases with increase of Re_λ, that means increasing of the clustering (as it was expected from qualitative observations).

To describe intermittency a running average of fluctuations of the dissipation rate $\varepsilon(t)$ of the velocity signal $u(t)$ can be used [13] (see also a discussion in [14])

$$\langle \delta \varepsilon_\tau^2 \rangle^{1/2} \sim \tau^{-\alpha_\varepsilon} \qquad (2)$$

where

$$\varepsilon_\tau = \frac{\int_0^\tau \varepsilon(t) dt}{\tau} \qquad (3),$$

and $\langle \delta \varepsilon_\tau^2 \rangle^{1/2}$ is standard deviation of ε_τ. Figure 3 shows an example of the scaling (2) providing the intermittency exponent α_ε for the wind-tunnel data (see [13] for more details and Fig. 5 for the statistical errors).

Events with high concentration of the zero-crossing points in the intermittent turbulent velocity signal provide main contribution to the $\langle \delta \varepsilon_\tau^2 \rangle$ due to high concentration of the statistically significant local maximums in these events [8]. Therefore, one can make use of statistical version of the theorem 'about mean' in order to estimate the standard deviation

$$\langle \delta \varepsilon_\tau^2 \rangle^{1/2} \sim \langle \delta \{ \max : u^2 \}_\tau^2 \rangle^{1/2} \langle \delta n_\tau^2 \rangle^{1/2} \qquad (4)$$

where $\{ \max : u^2 \}_\tau$ is maximum of $u(t)^2$ in the interval of length τ. Value of $\langle \delta \{ \max : u^2 \}_\tau^2 \rangle^{1/2}$ should grow with Re_λ (due to increasing of $\langle u^2 \rangle$) and, statistically, with the length of interval τ (the later growth is expected to be self-similar), i.e.

$$\langle \delta \{ \max : u^2 \}_\tau^2 \rangle^{1/2} = C(Re_\lambda) \cdot \tau^{\alpha_m} \qquad (5)$$

where constant $C(Re_\lambda)$ is a monotonically increasing function of Re_λ, and the statistical ('tail') exponent
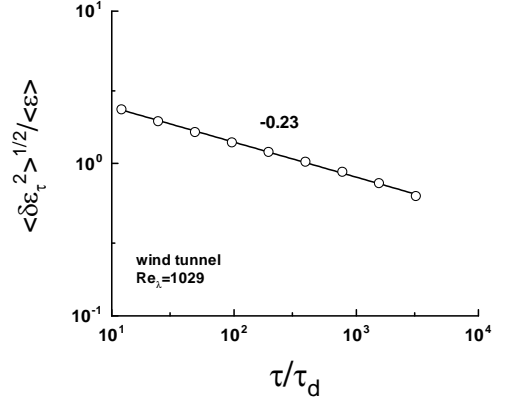

FIG. 3: Intermittency exponent as slope of the straight line for the longitudinal velocity signal obtained in the wind tunnel experiment [11] at $Re_\lambda = 1029$.
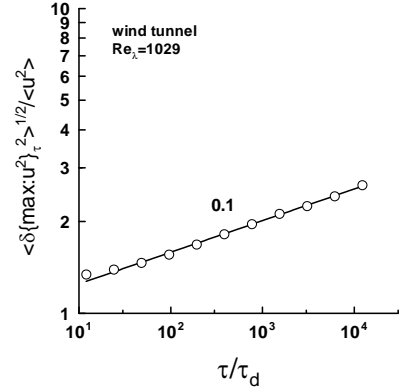

FIG. 4: Statistical ('tail') exponent $\alpha_m \simeq 0.1$ as slope of the straight line for the longitudinal velocity signal obtained in the wind tunnel experiment [11] at $Re_\lambda = 1029$.

$\alpha_m \geq 0$ is independent on Re_λ. Fig. 4 shows an example of scaling (5) providing the statistical ('tail') exponent $\alpha_m \simeq 0.10 \pm 0.01$. Substituting (5) into (4) and taking into account (1) and (2) we can infer a relation between the scaling exponents

$$\alpha_\varepsilon = \alpha_n - \alpha_m \qquad (6)$$

Unlike the exponent α_m the exponent α_n depends on Re_λ (see Fig. 2 and Ref. [4]). Therefore, we can learn from (6) that just the clustering of the high frequency velocity fluctuations (α_n in (6)) is responsible for dependence of the intermittency exponent α_ε on the Re_λ.
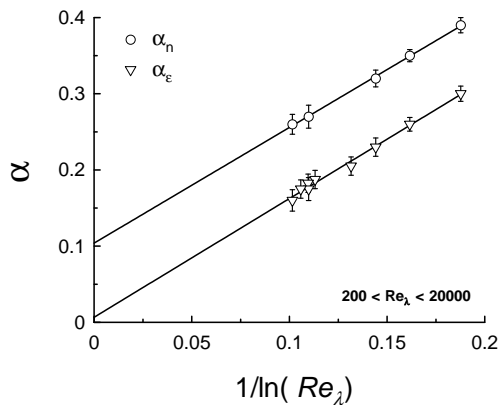
It is naturally consider the exponent functions through

FIG. 5: The cluster-exponent: α_n (circles), against $1/\ln Re_\lambda$ for velocity signal at different values of Reynolds number ($200 < Re_\lambda < 20000$). Scaling exponent α_ε (triangles) for fluctuations of the dissipation rate of the velocity signals.

$\ln R_\lambda$: $\alpha = \alpha(\ln Re_\lambda)$ [4],[15],[16] (see also Appendix). Following to the general idea of Ref. [16] (see also [4]) let us expend this function for large Re_λ in a power series:

$$\alpha(\ln Re_\lambda) = \alpha(\infty) + \frac{a_1}{\ln Re_\lambda} + \frac{a_2}{(\ln Re_\lambda)^2} + ... \quad (7)$$

In Figure 5 we show the values of α_n (circles) and α_ε (triangles) calculated for the velocity signals against $1/\ln Re_\lambda$ for $200 < Re_\lambda < 20000$ (the data are taken from the wind-tunnel experiment [11], from the atmospheric experiment [12], from a mixing layer and an atmospheric surface experiments [13]). The straight lines (the best fit) indicate approximation with the two first terms of the power series expansion (7)

$$\alpha_n(\ln Re_\lambda) \simeq 0.1 + \frac{3/2}{\ln Re_\lambda}, \qquad \alpha_\varepsilon(\ln Re_\lambda) \simeq \frac{3/2}{\ln Re_\lambda} \quad (8)$$

This means, in particular, that

$$\lim_{Re_\lambda \to \infty} \alpha_n \simeq 0.1, \qquad \lim_{Re_\lambda \to \infty} \alpha_\varepsilon \simeq 0 \quad (9)$$

The closeness of the constants $a_1 \simeq 3/2$ (the slopes of the straight lines in Fig. 5) in the approximations (8) of the

cluster-exponent α_n and of the intermittency exponent α_ε confirms the relationship (6) with $\alpha_m \simeq 0.1$ (cf Figs. 4 and 5).

Thus one can see that entire dependence of the intermittency exponent α_ε on Re_λ is uniquely determined by dependence of the cluster-exponent α_n on Re_λ. On the other hand, the cluster-exponent α_n itself is uniquely determined by energy spectrum of the velocity signal at suggestion that the velocity field is nearly Gaussian. This means that energy spectrum uniquely determines dependence of the intermittency exponent on the Reynolds number Re_λ. From conventional point of view on intermittency it seems as a paradox.

We use spectral simulation to illustrate the uniquely relationship between energy spectrum and cluster-exponent for the Gaussian signals; i.e. we generate Gaussian stochastic signal with energy spectrum given as data (the spectral data is taken from the original velocity signals). Obviously, the pure Gaussian signal obtained in such simulation does not possesses the turbulent intermittency properties resulting in the estimate (4). Figures 1b and 2b show, as examples, results of such simulation for the wind-tunnel velocity data (Fig. 1b) and for the atmospheric surface layer (Fig. 2b). Namely, Figures 1a and 2a show results of the calculations for the original velocity signals and Figs. 1b, and 2b show corresponding results obtained for the spectral simulations of the signals (to be sure, for the both Gaussian simulations $\alpha_\varepsilon \simeq 0.5$ unlike of α_n).

It is interesting to note that for isotropic turbulence with prominent inertial (Kolmogorov) range the only small-scale edge of the inertial range (this scale depends on Re_λ) determines value of the cluster-exponent α_n corresponding to the spectrum. I.e. role of the viscous dissipation in this respect is reduced to the effective small-scale 'cut-off' effect and just dependence of the 'cut-off' scale on Re_λ determines dependence of the intermittency exponent α_ε on Re_λ through equation (6) for such flows.

[1] G. Falkovich and K.R. Sreenivasan, Physics Today, **59**, 43 (2006).
[2] K.R. Sreenivasan and R.A. Antonia, Annu. Rev. Fluid Mech., **29**, 435 (1997).
[3] C.H. Gibson, Proc. Roy. Soc. London, Ser. A, **434**, 149 (1991).
[4] K.R. Sreenivasan and A. Bershadskii, J. Fluid Mech., **54** 477 (2006).
[5] K.R. Sreenivasan and A. Bershadskii, J. Stat. Phys., in press (2006).
[6] Monin A. S. and Yaglom A.M., 1975 Statistical Fluid Mechanics: Mechanics of Turbulence, V. 2 (The MIT Press, Cambridge).
[7] G. Stolovitzky, C. Meneveau and K.R. Sreenivasan, Phys. Rev. Lett., **80**, 3883 (1998).
[8] A. Bershadskii and A. Tsinober, Phys. Lett. A, **165** 37-41 (1992).
[9] G. Molchan, private communication.
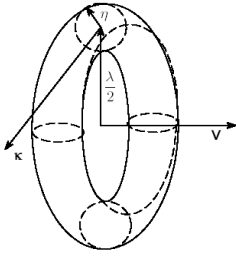[10] M.R. Leadbetter and J.D. Gryer, Bull. Amer. Math. Soc. **71**, 561 (1965).

FIG. 6: The vortex ring and its velocity.

[11] B.R. Pearson, P.-A. Krogstad and W. van de Water, Phys. Fluids, **14**, 1288 (2002).
[12] K.R. Sreenivasan and B. Dhruva, Prog. Theor. Phys. Supp., **130** 103 (1998).
[13] A. Praskovsky and S. Oncley, Fluid. Dyn. Res., **21**, 331 (1997).
[14] V.S. L'vov and I. Procaccia, Phys. Rev. Lett., **74**, 2690 (1995).
[15] B. Castaing, Y. Gagne and E.J. Hopfinger, Physica D, **46**, 177 (1990).
[16] G.I. Barenblatt and N. Goldenfeld, Phys. Fluids, **7**, 3078 (1995).
[17] G.K. Batchelor, An Introduction to Fluid Dynamics (Cambridge University Press, Cambridge, 1970).
[18] P.G. Saffman, Lectures on Homogeneous Turbulence. In *Topics in Nonlinear Physics* (ed. N.J. Zabusky), pp. 485-614, Springer-Verlag, New York, 1968.

I. APPENDIX

Thin vortex tubes (or filaments) are the most prominent hydrodynamical elements of turbulent flows [17]. The filaments are unstable in 3-dimensional space. In our recent paper [4] we investigated this instability in order to renormalize the Taylor microscale Reynolds number.

A straight line-vortex can readily develop a kink propagating along the filament with a constant speed. To estimate the velocity of propagation of such a kink let us first recall the properties of a ring vortex [17]. Its speed v is related to its diameter λ and strength Γ through

$$v = \frac{\Gamma}{2\pi\lambda} \ln\left(\frac{\lambda}{2\eta}\right), \qquad (A.1)$$

where η is the radius of the core of the ring and $\lambda/2\eta \gg 1$

(see figure 6).

If, for instance, a straight line-vortex develops a kink with a radius of curvature $\lambda/2$, then self-induction generates a velocity perpendicular to the plane of the kink. This velocity can be also calculated using (A.1). One can guess that in a turbulent environment, the most unstable mode of a vortex tube with a thin core of length L (integral scale) and radius η (Kolmogorov or viscous scale), will be of the order λ: Taylor-microscale [6],[18]. Then, the characteristic scale of velocity of the mode with the space scale λ can be estimated with help of equation (A.1). Noting that the Taylor-microscale Reynolds number is defined as [6]

$$Re_\lambda = \frac{v_0\lambda}{\nu}, \qquad (A.2)$$

where v_0 is the root-mean-square value of a component of velocity. It is clear that the velocity that is more relevant (at least for the processes related to the vortex instabilities) for the space scale λ is not v_0 but v given by (A.1). Therefore, corresponding effective Reynolds number should be obtained by the renormalization of the characteristic velocity in (A.2), as

$$Re_\lambda^{eff} = \frac{v\lambda}{\nu} \sim \frac{\Gamma}{2\pi\nu} \ln\left(\frac{\lambda}{2\eta}\right). \qquad (A.3)$$

It can be readily shown from the definition that

$$\frac{\lambda}{\eta} = const \; Re_\lambda^{1/2} \qquad (A.4)$$

where $const = 15^{1/4} \simeq 2$. Hence

$$Re_\lambda^{eff} \sim \frac{\Gamma}{4\pi\nu} \ln(Re_\lambda) \qquad (A.5)$$

The strength Γ can be estimated as

$$\Gamma \sim 2\pi v_\eta \eta \qquad (A.6)$$

where $v_\eta = \nu/\eta$ is the velocity scale for the Kolmogorov (or viscous) space scale η [6]. Substituting (A.6) into (A.5) we obtain

$$Re_\lambda^{eff} \sim \ln(Re_\lambda). \qquad (A.7)$$

Thus, for turbulence processes determined by the vortex instabilities the relevant dimensionless characteristic is $\ln(Re_\lambda)$ rather than Re_λ (cf Eq. (7) and Fig. 5).